Facets of the Finite Basis Problem for Finite Involution Semigroups

Igor Dolinka
University of Novi Sad, Serbia

An algebra $\mathbf{A}$ (a variety $\mathcal{V}$) is *nonfinitely based (NFB)* if its equational theory is not finitely axiomatizable; otherwise, $\mathbf{A}$ is *finitely based (FB)*. If $\mathbf{A}$ generates a locally finite variety, a stronger property of being *inherently nonfinitely based (INFB)* requires that any locally finite variety $\mathcal{V}$ containing $\mathbf{A}$ is NFB. Thus, if $\mathbf{A}$ and $\mathbf{B}$ are finite algebras, $\mathbf{A}$ is INFB, and $\mathbf{A}$ belongs to the variety generated by $\mathbf{B}$, then $\mathbf{B}$ must be (I)NFB, too. For a given class $\mathcal{C}$ of algebraic systems, *the finite basis problem (FBP)* asks for a characterization of (non)finitely based members of $\mathcal{C}$. Perhaps the most interesting case occurs when $\mathcal{C}$ consists of finite algebras.

As shown by R. McKenzie, the general question whether a finite algebra is NFB (known as *Tarski's finite basis problem*) is algorithmically unsolvable. On the other hand, the finite basis problem for finite algebras led to extensive and fruitful investigations in several particular algebraic theories, one of the most successful in this regard being *semigroup theory*. In this talk we review some recent (and several not so recent) results concerning the finite basis problem for finite *involution semigroups*. Recall that an involution semigroup is an algebra $(S, \cdot, {}^*)$ such that $(S, \cdot)$ is a semigroup and the identities $(xy)^* \approx y^*x^*$ and $(x^*)^* \approx x$ hold. Natural examples of involution semigroups include groups, inverse semigroups, and matrix semigroups equipped with various unary operations, such as the matrix transposition.

Somewhat surprisingly, the initial expectation that the additional, seemingly well-behaved unary operation cannot disturb the state of affairs holding for plain semigroups turned out to be rather far from the truth. There are several aspects of a number of universal-algebraic questions (including the FBP), where involution semigroups take on a life of their own. Among others, we plan to mention aspects of the FBP related to the following topics:

- the problem of characterizing INFB finite involution semigroups;
- matrix semigroups endowed with natural unary operations, such as the transposition, the Moore-Penrose inverse, and the operation of taking a symplectic transpose;
- various *partition monoids* with involution, such as the Brauer monoid and the annular monoid;
- power semigroups of finite groups;
- finite inverse semigroups;

etc. We also present some related results on involution semigroup varieties and point out several open questions.

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